ON EQUILIBRIA OF BID-ASK MARKETS

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ROBERT WILSON

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Robert Wilson * †

Among his many contributions to economic theory, Kenneth Arrow's studies of general equilibrium are especially important to the continuing development of the fine structure of market-mediated allocation processes. The paradigm of efficient decentralized allocation via market clearing prices developed from the Walrasian model in the long line of research given its greatest impetus by Arrow, Gerard Debreu, and their colleagues. The demonstration that 'perfectly' competitive complete markets, characterized by universal price-taking behavior, can in principle (absent non-convexities, etc.) attain an efficient allocation set the cornerstone of the theory of markets. By establishing the standard against which further studies of imperfectly competitive and incomplete markets are compared, this accomplishment continues to shape the agenda of continuing research on competitive processes.

Building on the foundation established by the theory of Walrasian models of general equilibrium with perfect competition, subsequent studies have aimed to elucidate the mechanisms of price formation in imperfectly competitive markets. One approach has been experimental and the data have strongly confirmed the predictive power of the Walrasian model, particularly in the case of pure exchange via publically announced bid and ask prices, even with relatively few participants. In replicated markets most of the gains from trade are realized

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and the trading prices converge quickly to or near the Walrasian price. A second approach has aimed to establish game-theoretic foundations for imperfectly competitive markets, focusing mainly on models of oligopoly and monopolistic competition but also recently on the extreme case of pure exchange represented by models of bilateral bargaining.

In this chapter we undertake an exploratory analysis of the connection between bilateral bargaining and the multilateral bargaining that is inherent in trading processes such as bid--ask markets. The theme suggested is that the theory of bilateral bargaining has a natural extension to multilateral trading in bid-ask markets, and then this extension generalizes naturally to encompass models of nearly perfect competition with many buyers and sellers. Our results are not complete, since we only examine certain necessary conditions and merely speculate on the appropriate sufficiency conditions, but nevertheless the internal consistency of the construction is encouraging.

Our analysis is based entirely on extrapolation from the important results obtained by Cramton [1984] for a model of bilateral bargaining between a buyer and a seller, with a crucial exception. Generalizing from the work of Rubinstein [1982] on a model with complete information, Cramton derives an equilibrium for a model with incomplete information in which each party's reservation price for the item to be traded is privately known. A main conclusion from this work is that the parties' impatience for an agreement, reflected in the interest rates they use to discount future payoffs, are major determinants of their bid, ask, and acceptance strategies. Moreover, trading takes time because each party needs to signal credibly his or her reservation price, and delay in offering or accepting proposed prices is an important credible signal. [1]

In generalizing Cramton's construction to multilateral markets, we propose to omit the role of impatience in the form of discounting in order to show how impatience arises endogenously from the competitive pressures among multiple buyers and among multiple sellers. That is, in multilateral situations a buyer or seller is impatient to trade lest a competitor usurp the opportunity with an earlier bid or ask that might be accepted. In doing so we obtain some

notational and analytical simplicity in an otherwise very complicated formulation, while at the same time we illustrate starkly the endogenous generation of impatience among the traders. A price we pay, however, is that the results are further incomplete due to the absence of a theory of the associated endgames that result when only one buyer or one seller remains who has not previously traded, and who is therefore not subject to competitive pressure. (The possible resolutions of this difficulty are discussed in §5.)

Showing that competitive pressure suffices to create impatience contributes to the second part of our agenda, which is to argue that bid--ask markets with many buyers and sellers approximate perfectly competitive markets; e.g., the equilibrium strategies imply near exhaustion of the gains from trade at prices approximating the Walrasian clearing price. This agenda requires that strong competitive pressures create overwhelming impatience so that trade proceeds quickly and little is lost in delay costs if in fact the participants have positive interest rates. One hopes to establish eventually, though not here, that with very many traders the market clears nearly immediately at prices predominately close to the Walrasian clearing price --- thereby establishing a game-theoretic foundation for the Walrasian model of perfect competition in the case of pure exchange with incomplete information. Such a construction would at the same time yield an answer to von Hayek's long-standing conundrum as to how it is that information dispersed among the traders is manifested in prices.

The execution of this agenda lies far beyond what will be accomplished in this chapter, and indeed its premises might still be found false. Nevertheless, here we take the first steps to construct the hypothesized form of the equilibrium.

1. Introduction

The bid-ask market is a common form of market organization, prevalent for instance in the commodity exchanges that are often used as the chief examples of perfectly competitive markets. It has been the subject of several experimen-

tal studies, some of which are reviewed by Plott [1982] and Smith [1982]. These experiments use the format of an oral double auction. In the simplest version, each trader is assigned a role as either a seller or a buyer, with an inelastic supply of or demand for one unit at a privately known reservation price. A limited duration is allowed in which at each time each trader can post an ask or a bid price, or accept one of the previously posted prices and thereby conclude a trade. A trader's payoff is the difference between his or her reservation price and the accepted price, or zero if the trader fails to trade. A prominent feature of the experimental results is that frequently most of the gains from trade are realized. Our aim is to show that this feature and others of interest would also obtain if the traders used strategies of the form derived by Cramton in the case of bilateral bargaining. Moreover, we will verify that such strategies satisfy at least the necessary conditions for an equilibrium.

The purpose of this chapter, then, is to construct a proposed equilibrium for a model of an oral double auction. The model used is somewhat stylized; e.g., time is taken to be continuous. Moreover, the equilibrium has the special form derived in prior work by Cramton [1984] for the special case of one seller and one buyer; no attempt is made here to find other equilibria. [2]

Our main result characterizes the proposed equilibrium as a multilateral sequential bargaining process in which sellers and buyers are endogenously paired off to trade until the gains from trade are nearly exhausted. If there are many sellers and buyers then there is a small chance that a profitable trade will be missed, and if so the unrealized gains from trade are small.

The price at which each pair trades is determined by the prospect of competition from other sellers and buyers. This feature differs from Cramton's characterization in which each trader's impatience to reach an agreement derives from an exogenously specified interest rate at which future payoffs are discounted.

The model is formulated in §2. [3] The proposed equilibrium is then described informally in §3 and its main implications are derived. In §4 we undertake a more detailed construction and specify the traders' strategies completely ex-

cept for contingencies far from the 'equilibrium path'. The special case of one seller or one buyer (i.e., an ordinary auction) is discussed briefly in §5 without specific results, since in this case it is necessary to add some exogenous source of impatience to sustain the form of the strategy of a monopolist trader. Concluding remarks are in §6.

A caution about terminology: we use the term 'equilibrium' in what follows even though we examine only whether the proposed equilibrium satisfies certain necessary conditions. The reader is urged to remember that it may yet be found that the construction fails because our assumptions or the necessary conditions prove to be insufficient.

2. Formulation

The following data of the game are common knowledge among the players. There are $m+n$ traders divided into m sellers and n buyers, and a single traded good to be exchanged for money. Each seller $i = 1,\ldots,m$ has an inelastic supply of one indivisible unit at any price not less than his privately known reservation price u_i, which we call his valuation. Similarly, each buyer $j = 1,\ldots,n$ has an inelastic demand for one unit at any price not exceeding his privately known valuation v_j. Thus, if seller i and buyer j trade at the price p then their payoffs are $p - u_i$ and $v_j - p$ respectively. The traders' preferences are linear in these payoffs; neither risk aversion nor wealth effects are present. The traders' valuations $U = (u_i)_{i=1,\ldots,m}$ and $V = (v_j)_{j=1,\ldots,n}$ are jointly distributed according to the distribution function

$$F(a,b) \equiv \Pr\{U \geq a \ \&\ V \leq b\} \tag{1}$$

with a positive density on the support $[u^*,1]^m \times [0,v^*]^n$. (Note that F is a right-cumulative distribution in terms of U; all other distribution functions used will be left cumulatives.) Suppose that $u^* < v^*$ so that gains from trade are not precluded. Assume that (U,V) are affiliated (cf. Milgrom and Weber [1982]), and symmetrically distributed in the sense that $F(a,b) = F(\alpha a, \beta b)$ for any two permutations α and β on m and n characters respectively. [4] Associated with F is the expectation operator $\mathcal{E}\{\cdot\}$ and the various

conditional expectation operators $\mathcal{E}\{\cdot \mid \cdot\}$; e.g., the one relevant for seller i is $\mathcal{E}\{\cdot \mid u_i\}$ which conditions on his valuation u_i. These operators can be conditioned further on observations of public events, interpreted for their informational significance using the traders' equilibrium strategies.

[Remark: A special case of such a distribution arises naturally in our construction and the reader may find it useful for interpretive purposes. Suppose that the buyers' valuations are independent of the sellers' valuations, and for illustration consider only the joint distribution of the buyers' valuations. Suppose that the buyers' valuations are conditionally independent and identically distributed given the value of a random variable $z \in [0, v^*]$, each with the distribution function F_B and the distribution function G for z. Unconditionally it is known only that the buyers' valuations do not exceed z, so that

$$\Pr\{V \le b\} = \int_0^{v^*} \prod_j F_B(\min\{b_j, z\})\, dG(z).$$

Such a case arises if the buyers' valuations were known initially to be independent but midway in the trading process it is inferred that those buyers who have not yet traded are those with valuations less than the ones who have already traded, but the minimum z among the valuations of those who have previously traded is not observed. Note that the distribution G is then the distribution of the rank order statistic appropriate to the number of buyers who have previously traded. See (5) below.]

As usual in game-theoretic formulations, the traders' equilibrium strategies are assumed to be common knowledge. The following trading rules are also common knowledge. Play is confined to a duration 1 and time, which runs continuously, is indexed by $t \in [0,1)$; since trade ceases at $t = 1$ there are no 'final offers'. At each time each trader who has not previously transacted can post an ask or a bid price or accept a posted price: sellers post asks and accept bids, buyers post bids and accept asks. Acceptance fixes that price for both the poster and the acceptor and these two traders are inactive thereafter; recontracting is excluded. All posted prices and acceptances are observed by

all traders, and each trader has perfect recall.

A strategy for a trader specifies at each time he is active (i.e., has not yet transacted) his action conditional on the common knowledge, his observations to date, and his valuation. His action is either his posted price, if any, or his acceptance of a posted price, thereby concluding his activity.

A belief system for a trader specifies at each time he is active a probability distribution over the other active traders' valuations, conditional on the common knowledge, his observations, and his valuation --- starting with F at time $t = 0$ conditioned on his valuation. An assignment of strategies and belief systems to the traders is consistent if the belief system is a conditional probability system satisfying Bayes' Rule wherever applicable (i.e., the most recent conditioning event is not null).[5]

We present a special kind of Nash equilibrium called a sequential equilibrium by Kreps and Wilson [1982]. A sequential equilibrium is a consistent assignment of strategies and beliefs to the traders such that for each trader, conditional on the time, his observations, and his valuation, his strategy for the remainder of the game is optimal given his current probability assessment and the other traders' strategies. We are interested, moreover, in a special kind of sequential equilibrium, called Markov perfect by Maskin and Tirole [1983], having the property that a trader's strategy at each contingency depends on the current history only through his current beliefs; that is, his current beliefs are a sufficient summary of the history. Interest in such an equilibrium stems from the fact that if other traders' strategies do not depend on the details of the prior history then a trader obtains no advantage from conditioning his strategy on these details. We shall also specify that the equilibrium is symmetric: all sellers use the same strategy and belief system, and similarly for buyers. Because their valuations differ, however, different sellers (or buyers) generally take different actions in similar contingencies.

Assume that each trader is indifferent as to the time he transacts; a trader does not discount future payoffs. We are thus able to construct the equilib-

rium so that it is independent of any particular interpretation of clock time. That is, it suffices to describe the strategies as implicit functions of time and then to allow that the actual equilibrium is obtained from any parameterization of time that is common knowledge among the traders; this technique is illustrated in §4. Since time runs continuously, moreover, the time index can be re-started at zero after each transaction. For example, if the first trade occurs at $t = 0.5$ then the remaining $m - 1$ sellers and $n - 1$ buyers enter a 'subgame' with a new clock that runs twice as fast as the original clock, so that both clocks register 1 and close the market at the same instant. (In practice there is a natural parameterization of time derived from the rate at which postings and acceptances are recorded; here we ignore this feature of the transaction technology.) The construction is designed in this way so that we need to describe only the strategies for a typical 'subgame' similar to the original game except for the number of traders and the conditioning of probability assessments.

Several technical specifications complete the formulation. First, we choose to exclude retention of posted prices: the posted prices at each time are those offered at that instant. Second, ties are resolved by some tie-breaking rule. The natural rule is a randomization: [a] multiple asks (or bids) at the same price at the same time are resolved by choosing one by an independent, uniform randomization; and [b] multiple acceptances of a single posted price are similarly chosen randomly. This natural rule complicates formulas, however, so here we opt in favor of the (admittedly impractical) rule of choosing the seller with the least valuation or the buyer with the greatest valuation. Third, multiple postings of different ask prices (or bid prices) at the same time are excluded: only the least ask (or the greatest bid) is posted. Similarly, an acceptance is interpreted as acceptance of the least current ask (or the greatest current bid); and posting, say, an ask less than or equal to the current posted bid is interpreted as acceptance. Further ambiguities can arise from the continuity of time and we resolve these by imagining that each instant is an infinitesimal interval that allows traders to respond to each other's ac-

tions: for example, a seller's strategy can specify in some contingency that he asks the minimum of his 'intended' price p and the least among the prices asked (if any) by competing sellers and not exceeding his valuation (e.g., if several sellers do this then the resulting posted price is the second lowest valuation, posted by the seller with the least valuation). As we shall see, these technical specifications have little effect on the particular equilibrium that is constructed.

Lastly we mention some notational conventions. The notation $x \sqcap y$ where x is a scalar and y is a vector indicates the vector $(\min\{x, y_j\})$, and similarly $x \sqcup y$ indicates the vector of maxima. Expectations and probability distributions that are conditioned on the publically observed history up to time t are denoted by $\mathcal{E}_t\{\cdot \mid \cdot\}$ and F_t. We use similar notation to represent conditioning on a seller's valuation [$F(\cdot \mid u_i)$] and a buyer's valuation [$F(\cdot \mid v_j)$]; no confusion should result. $\bar{F}_t(\bar{v})$ is used to indicate the distribution function of the maximum $\bar{v}$ among the buyers' valuations, and similarly for the distribution of the minimum of the sellers' valuations. Each distribution can be conditioned further on publically observed events or inferences from the traders' strategies. We use a semi-colon to separate conditioning on a valuation and conditioning on an extreme valuation; e.g., $\bar{F}_t(\bar{v} \mid u; \tilde{u})$ is the distribution function of $\bar{v}$ at time t conditioned on one seller's valuation being u and the minimum among the other sellers' valuations being $\tilde{u}$.

The exposition is eased by supposing that a trader who fails to trade in the game actually trades elsewhere at a reservation price equal to his valuation. Thus, the payoff of a seller with the valuation u can be interpreted as either his transaction price or, if he fails to trade, then his valuation u; similarly a buyer with the valuation v obtains either his transaction price or his valuation v. Following this convention, we use $\mathcal{U}(u)$ to indicate the expected price obtained by a seller (so in the original formulation his payoff is $\mathcal{U}(u) - u$), which he wants to maximize, and $\mathcal{V}(v)$ to indicate the expected price obtained by a buyer (so his payoff is $v - \mathcal{V}(v)$ in the original formulation), which he wants to minimize.

After two traders transact they become inactive and the remaining active traders enter an ensuing 'subgame': the typical notation uses F° to indicate the conditional probability distribution they carry into the ensuing 'subgame' (restricted to the support of the active traders' valuations conditioned on the history of observations, though this conditioning will not always be explicit), $\mathcal{U}^{\circ}$ and $\mathcal{V}^{\circ}$ to indicate the expected payoffs of a seller and a buyer in the ensuing subgame, etc. (This is a non-standard use of the term 'subgame', equivalent to the notion of a subform used by some authors to indicate what would be a subgame were it not for the effects of the incomplete information regarding the initial assignment of valuations to the traders. Here we induce a particular continuation game that the remaining active traders play by specifying their probability assessments as determined by conditioning on the prior history of play. However, later we shall allow that prior history can influence beliefs off the equilibrium path; see §4. Hereafter we use the term subgame without apology.)

3. The Equilibrium: Description and Implications

In this section we describe the equilibrium informally in order to convey its structural features, and then derive its main implications. The precise specification is deferred to §4.

The significant aspect of the equilibrium is that sellers and buyers are sequentially matched for transactions via an endogenous process that continues so long as there is a chance that gains from trade remain and time has not expired. Moreover, the price at which each pair transacts is determined by the competitive pressures from other sellers and buyers who are alternative trading partners. The following description sketches the workings of this process. We concentrate on the results of play according to the equilibrium strategies (i.e., along the 'equilibrium path') and mention only briefly the consequences of deviations, which will be elaborated in §4.

A key property of the equilibrium is that each strategy is a monotone function of the trader's privately known valuation. This property has several im-

portant consequences. First, given our choice of the tie-breaking rule (cf. §2), traders are matched in order of their valuations: the seller with the lowest valuation transacts first (if at all) with the buyer having the highest valuation. Consequently, each transaction moves the remaining active traders into an appropriately specified subgame that is similar to the original game; e.g., the numbers of active sellers and buyers are reduced from m and n to $m-1$ and $n-1$, time is again initialized at zero, and the remaining active traders' probability assessments are conditioned on the accumulated observations. Since the equilibrium is sequential, the strategies for the remaining active traders must constitute a sequential equilibrium for this subgame. By an induction argument, therefore, it suffices along the equilibrium path to describe the equilibrium for a typical subgame. However, the terminal subgames having only one seller or one buyer are special, because competitive pressure is absent, so their description is deferred to §5.

Second, in a typical subgame a price offered by one trader allows others to infer information about his valuation. The form of the equilibrium we construct is essentially characterized by this inferential process, which is derived from the prior work of Cramton [1984]. It is useful to distinguish between non-serious and serious ask and bid prices. An ask so high or a bid so low that it has zero probability of being accepted (according to the equilibrium strategies) is non-serious, and serious otherwise. For example, an ask is non-serious if it exceeds a posted ask or if there is no other serious ask offered and the ask exceeds the highest price that any buyer would accept according to the equilibrium strategies. We construct an equilibrium in which the traders' beliefs and strategies do not depend on the magnitudes of non-serious offers; similarly, not posting any offer is interpreted as the same as offering a non-serious price. Interest in such an equilibrium stems from the fact that a trader has no incentive to choose a particular non-serious offer from the many available unless it has some special significance as a signal: if other traders make no inferences from the numerical magnitudes of non-serious offers then a trader is indifferent as to which non-serious offer he makes. Since

we seek a Markov perfect equilibrium in which the details of the history have no special significance along the equilibirum path we want to exclude the possibility of signalling and thus rule out any inferential process that would motivate a trader to prefer one non-serious offer over another.

With this convention, the inferential process allows two possibilities after a trader makes an offer. The trader either makes a serious offer and the others then infer his valuation by inverting his strategy, or he makes a non-serious offer and the others infer only that his valuation is insufficient to prompt a serious offer. In the case of a seller, for example, if his ask price exceeds the maximum serious ask price then others infer only that his valuation exceeds the maximum valuation that would have led him to offer a serious price in that contingency. [Later we reinforce this approach further by requiring that also off the equilibrium path other traders infer that a trader's valuation is no less extreme than the most extreme that can be inferred from the entire history of his serious offers; for example, others infer that a seller's valuation is no more than the least valuation consistent, according to the equilibrium strategies, with any one of his serious ask prices.]

This inferential process has strong incentive effects, as we shall see. Each trader prefers to delay his first serious offer so as to prevent others from inferring that his valuation is more extreme than in fact it is. A seller, for example, wants to avoid others' inference that his valuation is lower than it actually is. He does this by delaying his first serious offer until it signals correctly his valuation --- and competitive pressure, the prospect that another trader will usurp the opportunity to trade, will assure that he wants to delay no longer.

This form of the equilibrium implies that a typical subgame divides into two phases. In the initial phase no serious offers are proposed and as time passes each trader continuously truncates the support of the probability distribution of the others' valuations as he infers that no trader has a valuation sufficient to induce a serious offer. We shall see that this initial phase continues until time expires only if the probability and magnitude of gains

from trade are both sufficiently small. Otherwise, it terminates with an initial serious offer that, by inference from the equilibrium strategies, reveals the posting trader's valuation. Since the offer is serious there is a positive probability that it is accepted immediately. If it is not accepted then the second phase ensues and the posting trading improves his offer until he obtains an acceptance or time expires.

Before describing the ensuing second phase it is useful to see the equilibrium strategies for the initial phase that are depicted schematically in Figure 1. The time t is represented along the abscissa and the traders' valuations are represented along the ordinate. Shown in the figure are the valuations $u^*(t)$ and $v^*(t)$ of the sellers and the buyers respectively that prompt their first serious offers $p(t)$ and $q(t)$ at time t; the difference between these is denoted by $\Delta(t) \equiv q(t) - p(t)$. Initially $u^*(0^+) \geq u^*$ and $v^*(0^+) \leq v^*$; one (or both) of these is an equality and the figure illustrates the case that $u^*(0^+) = u^*$ and $v^*(0^+) < v^*$. When time expires $u^*(1) \equiv \hat{u}$ and $v^*(1) \equiv \hat{v}$. For example, if a seller i has the valuation $u_i < u^*(t)$ then he makes his first serious offer before time t, but if $u_i \geq \hat{u}$ then he never makes a first serious offer in this subgame. As shown in the figure, $S(u)$ and $T(v)$ denote the times that a seller with the valuation u and a buyer with the valuation v make their first serious offers. At each time t the maximum serious ask and the minimum serious bid are $p(t)$ and $q(t)$ respectively. If, say, the serious ask price $p(t)$ is offered at time t by a seller with the valuation $u = u^*(t)$ then each buyer with a valuation in the interval between $v^{**}(t)$ and $v^*(t)$ proposes to accept; according to our tie-breaking rule the one with the highest valuation is selected for the transaction. Accordingly, the other traders infer the seller's valuation to be $u^*(t)$ and infer that it is a lower bound on the other sellers' valuations. If there is an immediate transaction then they infer that the buyer's valuation is between $v^{**}(t)$ and $v^*(t)$ and that it is an upper bound on the other buyers' valuations. Similarly, if the initial serious ask is not accepted then the traders infer that no buyer has a valuation exceeding $v^{**}(t)$. When time expires $p(1) = u^*(1) = v^{**}(1) = \hat{u}$ and $q(1) = v^*(1) = u^{**}(1) = \hat{v}$; e.g.,

if a seller with the valuation $\hat{u}$ asks $p(1^-) = \hat{u}$ just before time expires then it is accepted by any buyer with a higher valuation.

The fact that there is an interval of buyers' valuations implying willingness to accept a first serious ask price is another manifestation of a buyer's incentive to delay making a first serious offer. The benefit of delay is assurance that others will not infer that his valuation is too high, and the cost is the risk that another buyer will intervene earlier with a serious bid and capture an opportunity to trade; when this cost is removed buyers with valuations appreciably lower than the level prompting a serious bid are ready to accept. On the other hand, the fact that a seller about to conduct an auction among the buyers prefers to reveal his valuation is an instance of a general result due to Milgrom and Weber [1982; Theorems 17, 18].

Note that no serious offer occurs if no seller has a valuation below $\hat{u}$ and no buyer has a valuation above $\hat{v}$. In this case time expires without a transaction even though there is a positive probability that gains from trade are possible; however, the possible gain from a trade is bounded by the difference $\delta \equiv \hat{v} - \hat{u}$ and the probability is correspondingly small. That there is a positive difference between $\hat{v}$ and $\hat{u}$ is a necessary property of an equilibrium of this form since it is known that no trading rule can ensure that all gains from trade are realized; e.g., see Wilson [1983]. We shall see that it is also necessary that $\Delta(1) \equiv q(1) - p(1) \equiv \delta$ is positive.

The equilibrium strategies in the second phase are depicted schematically in Figure 2 for the case that a seller with the valuation u offered a first serious ask price $p(t)$ at time $t = S(u)$ that was not accepted. In the ensuing play this seller conducts a Dutch auction in which he continuously lowers his ask price $A(t;u)$ until it is accepted by some buyer, or his ask price declines to his valuation u as time expires and there is no trade. Initially $A(S(u);u) = p(S(u))$ and when time expires $A(1;u) = u$. Shown in the figure is the valuation $v^\circ(t;u)$ of the buyer who would accept at each time $t > S(u)$, starting with $v^\circ(S(u);u) = v^{**}(S(u))$ and ending at $v^\circ(1;u) = u$. As time passes without a transaction, therefore, the traders infer at time t that no buyer

has a valuation exceeding $v^{\circ}(t;u)$, and if the ask price $A(t;u)$ is accepted at time t then the remaining active traders infer that the accepting buyer's valuation is $v^{\circ}(t;u)$. Along the equilibrium path the other sellers make no serious offers and the probability assessment of their valuations remains unchanged. The fact that the two curves $A(t;u)$ and $v^{\circ}(t;u)$ coincide at the valuation u when time expires at time $t = 1$ is a general requirement of a sequential equilibrium: if the seller planned to keep his ask above his valuation then near the end he would want to cut his price to increase his chance of trading, and similarly for a buyer planning when to accept. Thus in this second phase failure to trade indicates an absence of gains from trade (this is consistent with the general theory since the seller's valuation has already been inferred by the other traders). [Off the equilibrium path the strategies in the Dutch auction are actually more complicated than is shown in the figure: if another seller intervenes with a lower ask price then play immediately reverts to a Dutch auction conducted by the intervening trader, whose valuation is presumed revealed by his offer by inference from the equilibrium strategies --- cf. §4.]

Figure 3 depicts schematically the analogous strategies in the case that the first serious offer was made by a buyer with the valuation v and not accepted.

In the following paragraphs we outline some of the consequences of an equilibrium of this form.

Probability Assessments

The inferential process in the two phases of a subgame is summarized as follows. As time passes without a serious offer in the initial phase the traders continuously truncate the support of the probability distribution to reflect the inference that no trader has a valuation sufficient to prompt a serious offer. Based on the publically observed history, therefore, the distribution function for the traders' valuations is

$$(2) \qquad F_t(a,b) = \frac{F(u^*(t) \sqcup a, v^*(t) \sqcap b)}{F(u^*(t)\mathbf{1}, v^*(t)\mathbf{1})}$$

at time t in the initial phase, where $\mathbf{1}$ denotes a vector of 1s. This is equivalent to the expectation operator $\mathcal{E}\{\cdot \mid \bar{u}\geq u^*(t), \bar{v}\leq v^*(t)\}$, where $\bar{u} = \min_i\{u_i \mid 1\leq i\leq m\}$ and $\bar{v} = \max_j\{v_j \mid 1\leq j\leq n\}$; of course, each trader further conditions on his valuation.

The first serious offer is interpreted as revealing precisely the valuation of the trader who proposes it. Taking the case of a seller, let $F(a,b \mid u)$ be the conditional distribution function given the seller's valuation u; then upon offering his first serious ask at time $t^* = S(u)$ the distribution function for the traders' valuations becomes

$$F_{t^*}(a,b) = \frac{F(u^*(t^*)\sqcup a, v^*(t^*)\sqcap b \mid u^*(t^*))}{F(u^*(t^*)\mathbf{1}, v^*(t^*)\mathbf{1} \mid u^*(t^*))}\,. \tag{3}$$

This is equivalent to the conditional expectation operator

$$\mathcal{E}\{\cdot \mid \bar{u}=u^*(t^*), \bar{v}\leq v^*(t^*)\}\,.$$

If the first serious offer is not accepted then again the traders truncate the support to reflect the inference that no trader has a valuation sufficient to induce acceptance according to the equilibrium strategies.

Similarly, as time passes without a transaction in the second phase the truncation reflects the inference that no trader has a valuation sufficient to accept the offer in the Dutch auction. At time $t > t^* = S(u)$, therefore, the distribution function for the traders' valuations becomes

$$F_t(a,b) = \frac{F(u^*(t^*)\sqcup a, v^\circ(t;u)\sqcap b \mid u^*(t^*))}{F(u^*(t^*)\mathbf{1}, v^\circ(t;u)\mathbf{1} \mid u^*(t^*))}\,. \tag{4}$$

This is equivalent to the conditional expectation operator

$$\mathcal{E}\{\cdot \mid \bar{u}=u^*(t^*), \bar{v}\leq v^\circ(t;u)\}\,.$$

If a transaction occurs then the remaining active traders move into a subgame with a probability assessment that depends on the circumstance.

If the first serious ask was accepted then the ensuing subgame is initialized with the probability assessment

$$F^\circ(a,b) = \int_{v^{**}(t^*)}^{v^*(t^*)} \left[\frac{F_{t^*}(a, \bar{v}\sqcap b \mid \bar{v})}{F_{t^*}(u^*(t^*)\mathbf{1}, v^*(t^*)\mathbf{1} \mid \bar{v})}\right] d\bar{F}_{t^*}(\bar{v})/[1-\bar{F}_{t^*}(v^{**}(t^*))] \tag{5}$$

on the support $[u^*(t^*),1]^{m-1} \times [0,v^*(t^*)]^{n-1}$ for the remaining active traders; here $F_t(a,b \mid v)$ represents the further conditioning of F_t on a buyer's valuation v, and $\bar{F}_t(\bar{v})$ is the marginal distribution (derived from F_t) of the maximum valuation $\bar{v}$ among the buyers. This is equivalent to the conditional expectation operator

$$\mathcal{E}\{\cdot \mid \bar{u}{=}u^*(t^*), \bar{v} \in [v^{**}(t^*), v^*(t^*)]\}.$$

If the seller's later ask in the Dutch auction at time $t > S(u)$ is accepted then the ensuing subgame is initialized with the distribution

(6) $$F^\circ(a,b) = \frac{F_t(a, v^\circ(t;u) \sqcap b \mid v^\circ(t;u))}{F_t(u^*(t^*)\mathbf{1}, v^\circ(t;u)\mathbf{1})}$$

on the support $[u^*(t^*),1]^{m-1} \times [0,v^\circ(t;u)]^{n-1}$ for the remaining active traders. This is equivalent to the conditional expectation operator

$$\mathcal{E}\{\cdot \mid \bar{u}{=}u^*(t^*), \bar{v}{=}v^\circ(t;u)\}.$$

In either case, the distribution F° that initializes the ensuing subgame satisfies the requirement that the subgame is similar to the original game: the remaining active traders' valuations are affiliated and symmetrically distributed (Milgrom and Weber [1982]).

Subgame Payoffs

The implications of these strategies for the expected payoffs obtained by a trader, say a seller i, are depicted in Figures 4, 5, and 6 for three ranges in which his valuation u can lie. Figure 4 represents the situation if $u < \hat{u}$ so that he plans to offer a serious ask price at the time $S(u)$ if no other trader does so earlier. The abscissa $\tilde{u}$ represents the minimum among the other sellers' valuations and the ordinate $\bar{v}$ represents the maximum among the buyers' valuations. The three regions shown correspond to the three cases that there is no trade in this subgame (his payoff is his valuation u), some other seller trades with a buyer (his payoff is the expected payoff U° in the ensuing subgame, depending via F° on $\tilde{u}$ and $\bar{v}$ for the transaction that occurs), and the

case that he trades with the buyer having the valuation $\bar{v}$ at a price $P(u,\bar{v})$. This price is defined by

$$(7) \qquad P(u,v) \equiv \begin{cases} p(S(u)) & \text{if } S(u) < T(v)\ \&\ v \in [v^{**}(S(u)), v^{*}(S(u))], \\ A(t;u) & \text{if } S(u) < T(v)\ \&\ v = v^{\circ}(t;u) < v^{**}(S(u)), \\ q(T(v)) & \text{if } S(u) > T(v)\ \&\ u \in [u^{*}(T(v)), u^{**}(T(v))], \\ B(t;v) & \text{if } S(u) > T(v)\ \&\ u = u^{\circ}(t;v) > u^{**}(T(v)). \end{cases}$$

Figure 7 shows how the transaction price $P(u,v)$ varies with v. The notation $A(u,v) \equiv A((v^{\circ})^{-1}(v);u)$ and $B(u,v) = B((u^{\circ})^{-1}(u);v)$ is used. Note the discontinuity where $S(u) = T(v)$, or equivalently $(u,v) = (u^{*}(t), v^{*}(t))$.

As shown in Figures 5 and 6, these regions are more complicated when the seller is uncertain that a serious offer will be made, since there is a small triangular region in which gains from trade are missed. Table 1 summarizes the features shown in Figures 4, 5, and 6. An analogous table describes a buyer's contingent subgame payoffs.

Table 1: A Seller's Contingent Subgame Payoffs

	$\bar{v} \le (u \sqcap \tilde{u}) \sqcap \hat{v}$	$\bar{v} > (u \sqcap \tilde{u}) \sqcap \hat{v}$ $\bar{v} \le (u \sqcap \tilde{u}) \sqcup \hat{v}$	$\bar{v} > (u \sqcap \tilde{u}) \sqcup \hat{v}$
$\tilde{u} < u \sqcap \hat{u}$	u	$\mathcal{U}^{\circ}$	$\mathcal{U}^{\circ}$
$u \sqcap \hat{u} \le \tilde{u} < u$	u	u	$\mathcal{U}^{\circ}$
$u < \tilde{u}$	u	$\begin{Bmatrix} u & \text{if } u \ge \hat{u} \\ P(u,\bar{v}) & \text{if } u < \hat{u} \end{Bmatrix}$	$P(u,\bar{v})$

Using this description of a seller's contingent subgame payoffs, we develop a general formula for the expected subgame payoff of a seller. At time t let $\bar{F}_t(\bar{v} \mid w;\tilde{u})$ be the distribution function of the maximum $\bar{v}$ among the buyers' valuations conditional on one seller's valuation being w and the minimum among the other sellers' valuations being $\tilde{u}$; and let $\tilde{F}_t(\tilde{u} \mid w)$ be the conditional distribution function of the latter. Also, let $\mathcal{U}^{\circ}(u \mid \tilde{u},\bar{v})$ be the expected payoff of a seller with the valuation u if he remains active in the ensuing subgame after a transaction between another seller with the valuation

$\tilde{u}$ and a buyer with the valuation $\bar{v}$. The dependence of $\mathcal{U}^\circ$ on $(\tilde{u},\bar{v})$ is via its dependence on F° as in (5) and (6). Corresponding to the three rows of Table 1, define

$$
(8) \qquad
\begin{aligned}
I_t^1(u \mid w;\tilde{u}) &= \int_0^{\tilde{u}} u\, d\bar{F}_t(\bar{v} \mid w;\tilde{u}) + \int_{\tilde{u}}^{v^*(t)} \mathcal{U}^\circ(u \mid \tilde{u},\bar{v})\, d\bar{F}_t(\bar{v} \mid w;\tilde{u}),\\
I_t^2(u \mid w;\tilde{u}) &= \int_0^{\hat{v}\sqcup\tilde{u}} u\, d\bar{F}_t(\bar{v} \mid w;\tilde{u}) + \int_{\hat{v}\sqcup\tilde{u}}^{v^*(t)} \mathcal{U}^\circ(u \mid \tilde{u},\bar{v})\, d\bar{F}_t(\bar{v} \mid w;\tilde{u}),\\
I_t^3(u,\check{u} \mid w;\tilde{u}) &= \int_0^{\check{u}+\xi(\check{u})} u\, d\bar{F}_t(\bar{v} \mid w;\tilde{u}) + \int_{\check{u}+\xi(\check{u})}^{v^*(t)} P(\check{u},\bar{v})\, d\bar{F}_t(\bar{v} \mid w;\tilde{u}),
\end{aligned}
$$

where

$$
(9) \qquad \xi(u) \equiv \begin{cases} \hat{v} - u & \text{if } \hat{u} \le u \le \hat{v}, \\ 0 & \text{otherwise} \end{cases}
$$

Observe that this notation distinguishes between the seller's actual valuation u, the valuation w upon which he conditions his probability assessment, and the valuation $\check{u}$ upon which he conditions his strategy, although in equilibrium these must all be identical.

With this notation we specify a general formula that is useful later. The seller's expected subgame payoff if he conditions his probability assessment on w and his strategy on $\check{u}$ is

$$
(10) \qquad
\begin{aligned}
\check{\mathcal{U}}_t(u,\check{u} \mid w) = &\int_{u^*(t)}^{\hat{u}\sqcap\check{u}} I_t^1(u \mid w;\tilde{u})\, d\tilde{F}_t(\tilde{u} \mid w) + \int_{\hat{u}\sqcap\check{u}}^{\check{u}} I_t^2(u \mid w;\tilde{u})\, d\tilde{F}_t(\tilde{u} \mid w)\\
&+ \int_{\check{u}}^{1} I_t^3(u,\check{u} \mid w;\tilde{u})\, d\tilde{F}_t(\tilde{u} \mid w),
\end{aligned}
$$

at each time $t < S(u)$ before any serious offer. In equilibrium, of course, the actual expected subgame payoff is $\mathcal{U}(u) = \check{\mathcal{U}}_0(u,u \mid u)$. A similar formula can be constructed for a buyer.

It is important to note that potentially there could be a discontinuity in $\mathcal{U}(u)$ at $u = \hat{u}$ where ξ is discontinuous. In fact, however, continuity is assured since $P(\hat{u},\bar{v}) = \hat{u}$ for all $\bar{v} \in [\hat{u},\hat{v}]$ due to the specifications $p(1) = \hat{u}$ and $v^{**}(1) = \hat{u}$ for the equilibrium strategies. That is, since continuity is generally necessary, the positivity of the difference $\Delta(1) \equiv q(1) - p(1)$ is a corollary of the positivity of the difference $\delta \equiv \hat{v} - \hat{u}$, and indeed $\Delta(1) = \delta$.

The Revelation Game

A crucial test of whether the specified strategies and beliefs are a sequential equilibrium is obtained by analyzing the 'revelation games' they induce. At each time t the induced revelation game allows a trader the option of selecting the valuation on which his strategy is conditioned. A seller, for example, can condition his strategy on any valuation $\breve{u}$, possibly different than his actual valuation u. An equilibrium in the original game necessarily has the property that each trader prefers to condition his strategy on his actual valuation, since otherwise in the original game he would have preferred a different strategy. In the next paragraphs we derive the necessary conditions implied by this requirement. We consider only the case of a seller with the actual valuation u; the case of a buyer is analogous.

First consider the situation at a time $t > S(u)$ in the second phase after the seller has made the first serious offer. Using the notational scheme introduced above, the seller's expected payoff according to the specified strategies is

$$(11) \qquad \breve{U}_t(u, \breve{u} \mid w) = \int_0^{\breve{u}} u \, d\bar{F}_t(\bar{v} \mid w) + \int_{\breve{u}}^{v^\circ(t;u)} P(\breve{u}, \bar{v}) \, d\bar{F}_t(\bar{v} \mid w) \, ,$$

if he conditions his subsequent strategy on $\breve{u}$ and his beliefs on w. At this time, of course, he is conducting a Dutch auction with

$$(12) \qquad P(\breve{u}, \bar{v}) = A(\breve{u}, \bar{v}) \equiv A((v^\circ)^{-1}(\bar{v}); \breve{u}) \, ;$$

moreover, in this case $P(\breve{u}, \bar{v})$ is an increasing function of $\breve{u}$ and $P(\breve{u}, \breve{u}) = \breve{u}$. From these two properties it follows that $\breve{U}_t(u, \breve{u} \mid u)$ is maximized by the choice $\breve{u} = u$ as required; e.g., $\partial \breve{U}_t(u, \breve{u} \mid u)/\partial \breve{u} = 0$ at $\breve{u} = u$.

Second, consider the situation at a time $t > T(\bar{v})$ in the second phase after the first serious offer has been made by the buyer with the valuation $\bar{v}$, and assume that both u and $\breve{u}$ are less than $\bar{v}$. According to the specified strategies the seller's expected subgame payoff, conditional on the maximum of the buyers' valuations being $\bar{v}$, is

$$(13) \qquad \breve{U}(u, \breve{u} \mid w; \bar{v}) = \int_{u^*(t)}^{\breve{u}} U^\circ(u \mid \tilde{u}, \bar{v}) \, d\tilde{F}_t(\tilde{u} \mid w; \bar{v}) + \int_{\breve{u}}^{1} P(\breve{u}, \bar{v}) \, d\tilde{F}_t(\tilde{u} \mid w; \bar{v}) \, ,$$

if he conditions his subsequent strategy in this subgame on $\check{u}$ and his beliefs on w. In this case, of course, the buyer is conducting an ascending Dutch auction with

$$(14) \qquad P(\check{u},\bar{v}) = B(\check{u},\bar{v}) \equiv B((u^\circ)^{-1}(\check{u});\bar{v});$$

moreover, $P(\check{u},\bar{v})$ is an increasing function of $\check{u}$ and $P(\bar{v},\bar{v}) = \bar{v}$. A necessary condition for the requirement that $\check{\mathcal{U}}_t(u,\check{u} \mid u)$ is maximized by the choice $\check{u} = u$, is therefore that the corresponding derivative is zero:

$$(15) \qquad 0 = [\mathcal{U}^\circ(u \mid u,\bar{v}) - B(u,\bar{v})]\tilde{F}_t'(u \mid u;\bar{v}) + \frac{\partial B(u,\bar{v})}{\partial u}\int_u^1 d\tilde{F}_t(\tilde{u} \mid u;\bar{v}).$$

The seeming dependence of this condition on the time t can be eliminated by expressing it in terms of the 'hazard rate'

$$(16) \qquad \phi(u;\bar{v}) \equiv \tilde{F}_t'(u \mid u;\bar{v})\Big/\int_u^1 d\tilde{F}_t(\tilde{u} \mid u;\bar{v}),$$

so as to cancel out the common proportionality factor; cf. (4). Thus, the relevant condition is

$$(17) \qquad 0 = [\mathcal{U}^\circ(u \mid u,\bar{v}) - B(u,\bar{v})]\phi(u;\bar{v}) + \frac{\partial B(u,\bar{v})}{\partial u}.$$

Invoking the boundary condition $B(\bar{v},\bar{v}) = \bar{v}$ mentioned earlier, this implies that

$$(18) \qquad B(u,\bar{v}) = \frac{\bar{v}\Phi(\bar{v};\bar{v}) - \int_u^{\bar{v}} \mathcal{U}^\circ(x \mid x;\bar{v})\,d\Phi(x;\bar{v})}{\Phi(u;\bar{v})},$$

where $\Phi(u;\bar{v}) \equiv \exp\{-\int_{u^*}^{u} \phi(x;\bar{v})\,dx\}$. For example, if the sellers' valuations happen to be independent with the distribution function G then $\phi(u) = [m-1]G'(u)/[1-G(u)]$ and $\Phi(u) = [1-G(u)]^{m-1}$. In general, if there are many sellers then one with the valuation u accepts a bid close to his continuation value $\mathcal{U}^\circ(u \mid u,\bar{v})$ in an ensuing subgame.

Similarly, a parallel analysis yields a differential equation for $A(\bar{u},v)$ that determines the traders' strategies during a seller's descending Dutch auction:

$$(19) \qquad 0 = [A(\bar{u},v) - \mathcal{V}^\circ(v \mid v,\bar{u})]\psi(v;\bar{u}) + \frac{\partial A(\bar{u},v)}{\partial v},$$

where

$$\psi(v;\bar{u}) \equiv \tilde{F}_t'(v \mid v;\bar{u}) / \int_0^v d\tilde{F}_t(\tilde{v} \mid v;\bar{u}), \tag{20}$$

using an obvious transposition of notation. Invoking the boundary condition $A(\bar{u},\bar{u}) = \bar{u}$, this implies that

$$A(\bar{u},v) = \frac{\bar{u}\Psi(\bar{u};\bar{u}) + \int_{\bar{u}}^{v} \mathcal{V}^\circ(y \mid y,\bar{u})\, d\Psi(y;\bar{u})}{\Psi(v;\bar{u})}, \tag{21}$$

where $\Psi(v;\bar{u}) \equiv \exp\{-\int_v^{v^*} \psi(y;\bar{u})\,dy\}$. Again, if there are many buyers then the hazard rate is large and a buyer accepts an ask close to his continuation value $\mathcal{V}^\circ(v \mid v,\bar{u})$.

It will be mandatory in the next section that we verify that $A(\bar{u},v)$ and $B(u,\bar{v})$ satisfy these two relationships. We show, in fact, that these relationships completely characterize the traders' strategies along the equilibrium path during the second phase.

Lastly, consider the situation at a time $t \leq S(u)$ in the initial phase before any serious offers have been made. Assume that both u and $\check{u}$ are less than $\hat{u}$, and in particular $\xi(\check{u}) = 0$. According to the specified strategies, the seller's expected payoff is then

$$\check{\mathcal{U}}_t(u,\check{u} \mid w) = \int_{u^*(t)}^{\check{u}} I_t^1(u \mid w;\tilde{u})\, d\tilde{F}_t(\tilde{u} \mid w) + \int_{\check{u}}^{1} I_t^3(u,\check{u} \mid w;\tilde{u})\, d\tilde{F}_t(\tilde{u} \mid w), \tag{22}$$

if he conditions his strategy on $\check{u}$ and his beliefs on w. A necessary condition for the requirement that $\check{\mathcal{U}}(u,\check{u} \mid u)$ is maximized by the choice $\check{u} = u$ is therefore that the corresponding derivative is zero (and decreasing):

(23)

$$\begin{aligned}
0 &= [I_t^1(u \mid u;u) - I_t^3(u,u \mid u;u)]\tilde{F}_t'(u \mid u) + \frac{\partial}{\partial \check{u}} \int_u^1 I_t^3(u,\check{u} \mid u;\tilde{u})\, d\tilde{F}_t(\tilde{u} \mid u), \\
&= \int_u^{v^*(t)} [\mathcal{U}^\circ(u \mid u;\bar{v}) - P(u,\bar{v})]\, d\bar{F}_t(\bar{v} \mid u;\tilde{u}=u) \cdot \tilde{F}_t'(u \mid u) \\
&\quad + \int_u^1 \left\{ \int_u^{v^*(t)} \frac{\partial P(u,\bar{v})}{\partial u}\, d\bar{F}_t(\bar{v} \mid u;\tilde{u}) + \Delta(S(u))\bar{F}_t'(v^*(S(u)) \mid u;\tilde{u}) \right\} d\tilde{F}_t(\tilde{u} \mid u), \\
&= \int_u^{v^*(t)} \left\{ [\mathcal{U}^\circ(u \mid u;\bar{v}) - P(u,\bar{v})]\tilde{F}_t'(u \mid u;\bar{v}) + \frac{\partial P(u,\bar{v})}{\partial u}[1 - \tilde{F}_t(u \mid u;\bar{v})] \right\} d\bar{F}_t(\bar{v} \mid u) \\
&\quad + \Delta(S(u))[1 - \tilde{F}_t(u \mid u;v^*(S(u)))]\bar{F}_t'(v^*(S(u)) \mid u).
\end{aligned}$$

Here, the second equality again uses the property that $P(u,u) = u$; also recall that $\Delta(t) \equiv q(t) - p(t)$ is the jump discontinuity in P at $t = S(u) = T(\bar{v})$. The third equality uses the identity

$$\bar{F}_t'(\bar{v} \mid u;\tilde{u})\tilde{F}_t'(\tilde{u} \mid u) \equiv \tilde{F}_t'(\tilde{u} \mid u;\bar{v})\bar{F}_t'(\bar{v} \mid u),$$

according to the rules of conditional probability.

The condition (23) must hold at all times $t \leq S(u)$ but note that it is sufficient that it holds where $u^{**}(t) \geq u$ since at earlier times (15) assures that the integrand within the curly brackets in (23) is zero for $\bar{v} > v^*((u^{**})^{-1}(u))$; that is, in this region the buyer with the maximum valuation makes the first serious offer and the seller does not accept it, so an ascending Dutch auction ensues.

At the time $t = S(u)$ when the seller makes his first serious offer we have $\tilde{F}_t(u \mid u;\bar{v}) = 0$ and $\tilde{F}_t'(u \mid u;\bar{v}) = \phi(u;\bar{v})$; consequently a special case of (23) is

$$(24)\qquad \begin{aligned} 0 = \int_u^{v^*(S(u))} & \left\{ [\mathcal{U}^\circ(u \mid u;\bar{v}) - P(u,\bar{v})]\phi(u;\bar{v}) + \frac{\partial P(u;\bar{v})}{\partial u} \right\} d\bar{F}_{S(u)}(\bar{v} \mid u;u) \\ & + \Delta(S(u))\bar{F}_{S(u)}'(v^*(S(u)) \mid u;u). \end{aligned}$$

At this time $\bar{F}_{S(u)}$ is conditioned on $\tilde{u} \geq u$ (indicated by the semi-colon) by inference from the other sellers' strategies. [6] Also $\partial P(u;\bar{v})/\partial u$ is properly interpreted here as the derivative from the left, and in particular in the relevant range of $\bar{v}$:

$$(25)\qquad P(u,\bar{v}) = \begin{cases} A(u, v^\circ(S(u);u)) & \text{if } v^\circ(S(u);u) \leq \bar{v} \leq v^*(S(u)), \\ A(u,\bar{v}) & \text{if } u < \bar{v} < v^\circ(S(u);u). \end{cases}$$

The condition (24) essentially determines the seller's planned time $S(u)$ of his first serious offer. Alternatively, it can be interpreted as determining the upper bound $v^*(S(u))$ of the support of the buyers' valuations that prompts the seller to make his first serious offer. If there are many sellers so that the hazard rate ϕ is large, then a seller plans to make his first serious offer early. If (24) holds then also (23) does since it corresponds to an expectation of (24). It will be mandatory in the next section that we verify that these conditions, and their analogues for a buyer, are satisfied.

An examination of the various conditions derived above for the revelation game reveals that there is one less equation than is required to determine all the functions entering the specification of the equilibrium strategies. Later we show that the missing condition is that $v^{*\prime}(t) = u^{*\prime}(t)$ at all times $t \in (0,1)$; that is, the supports of the buyers' and sellers' valuations contract at the same rate during the initial phase.

Monotonicity

Each of the formulas (10), (11), (13), and (22) for a seller's expected payoff have the property that $\partial \breve{\mathcal{U}}_t/\partial u$ is precisely the seller's conditional probability that he fails to trade (using an evident induction on the subgames): this is necessarily so since u is his payoff if he fails to trade. Additionally, the revelation conditions associated with each one ensure that $\partial \breve{\mathcal{U}}_t/\partial \breve{u} = 0$ at $\breve{u} = u$. Finally, affiliation implies that $\partial \breve{\mathcal{U}}_t/\partial w \geq 0$. Combining these results yields the requisite property that $d\mathcal{U}(u)/du \geq 0$; that is, a seller's expected payoff is a nondecreasing function of his valuation u. The analogous property holds for buyers as well.

The various formulas also involve a seller's continuation value $\mathcal{U}^{\circ}(u \mid \tilde{u}, \bar{v})$ in an ensuing subgame. Again, affiliation implies that $\partial \mathcal{U}^{\circ}/\partial \tilde{u} \geq 0$. For example, $\mathcal{U}^{\circ}(u \mid u, \bar{v})$ as in (15) or (24) is a non-decreasing function of the seller's valuation u, and the analogous property holds for a buyer.

Inefficiency

The condition (24) gives the illusion that it is possible that all gains from trade are realized, as would be the case if $v^*(1) = \hat{u}$ and $\Delta(1) = 0$ at the close of the market. This is not possible, however, since it is the larger of two roots of (24) that corresponds to the optimal choice for the seller. To see this, interpret (24) as the condition that determines the seller's optimal choice of $v^*(u) \equiv v^*(S(u))$, in which case the second-order condition requires that the first term on the right of (24) is decreasing. If $v^*(\hat{u}) = \hat{u}$, however,

the derivative of this term with respect to $v^*(u)$ at $\hat{u}$ is

$$(26) \qquad \left\{[U^\circ(\hat{u} \mid \hat{u};\hat{u}) - P(\hat{u},\hat{u})]\phi(\hat{u};\hat{u}) + \frac{\partial P(\hat{u};\hat{u})}{\partial u}\right\}\bar{F}_1'(\hat{u} \mid \hat{u}) \geq 0\,,$$

since $U^\circ(\hat{u} \mid \hat{u};\hat{u}) = P(\hat{u},\hat{u}) = \hat{u}$ and $\partial P(\hat{u};\hat{u})/\partial u \geq 0$.

This concludes our description of the equilibrium (along the equilibrium path) and its main consequences. In the next section we construct the equilibrium strategies as solutions to the traders' personal optimization problems, given that each anticipates that all other traders will be using the specified strategies.

4. The Equilibrium: Construction

We divide the construction between characterization of the equilibrium path and analysis of off-the-equilibrium-path behavior. In the first part we are mainly concerned with establishing that the characterizations derived in the analysis of the revelation games are valid. We concentrate on the necessary conditions that specify formulas for the strategies, but make some references to the sufficient conditions. In the second part we delineate off-the- equilibrium-path beliefs that are sufficient to support the equilibrium by deterring deviations.

It is sufficient to characterize only those features of the strategies that are independent of the parameterization of time, since the remainder can then be determined once a particular parameterization has been selected. For the second phase of a subgame we characterize the price $A(u,\bar{v})$ at which a seller making the first serious offer subsequently trades with a buyer having the highest valuation $\bar{v} > u$. In this case the parameterization of time can be taken to be the specification of the valuation $v^\circ(t;u)$ of the buyer accepting the ask price $A(t;u)$, subject to the conditions that v° is a declining function of time and that $v^\circ(1;u) = u$; thus, knowing the parameterization we obtain $A(t;u) = A(u, v^\circ(t;u))$. Alternatively, one could specify the temporal sequence $A(\cdot;u)$ of declining ask prices subject to $A(1;u) = u$ and then derive $v^\circ(t;u)$. Similarly, we characterize the price $B(\bar{u},v)$ at which a buyer making the first

serious offer subsequently trades with the seller having the lowest valuation $\bar{u} < v$, and then $B(t;v) = B(u^\circ(t;v),v)$. Knowing these functions also enables us to specify that if $S(u) = t = T(v)$ then $p(t) = A(t;u)$ and $q(t) = B(t;v)$, and also $u^{**}(t) = u^\circ(t;v)$ and $v^{**}(t) = v^\circ(t;u)$. For the first phase, therefore, it remains only to characterize the times $S(u)$ and $T(v)$ of the first serious offers of a seller and a buyer with the valuations $u < \hat{u}$ and $v > \hat{v}$.

Strategies in the Dutch Auctions

We study the ascending Dutch auction by a buyer in a market with more than one seller ($m > 1$); a descending Dutch auction by a seller is analogous.

On the Equilibrium Path

A seller with the valuation u anticipates the sequence $\{B(t;\bar{v}) \mid T(\bar{v})<t<1\}$, of bid prices after the first serious offer by a buyer inferred to have the valuation $\bar{v} > u$, and that any other seller with a valuation less than $u^\circ(t;\bar{v})$ will accept any bid price above $B(t;\bar{v})$. At a time t this seller prefers to wait a further interval $\epsilon > 0$ if the current bid is less than the expectation of the price he can obtain by waiting. This subsequent price is either his expected price in an ensuing subgame, with the probability

$$(27) \qquad Q_t(\epsilon \mid u,\bar{v}) \equiv \int_{u^\circ(t;\bar{v})}^{u^\circ(t+\epsilon;\bar{v})} d\tilde{F}_t(\tilde{u} \mid u;\bar{v})$$

that another seller accepts in the interim, or the subsequent bid $B(t+\epsilon;\bar{v})$ with the complementary probability. That is, he prefers to wait if there exists $\epsilon > 0$ such that

$$(28) \qquad \begin{aligned} B(t;\bar{v}) <\, & \mathcal{E}_t\{\mathcal{U}^\circ(u \mid \tilde{u},\bar{v}) \mid u^\circ(t;\bar{v})<\tilde{u}<u^\circ(t+\epsilon;\bar{v})\}Q_t(\epsilon \mid u,\bar{v}) \\ & + B(t+\epsilon;\bar{v})[1 - Q_t(\epsilon \mid u,\bar{v})]. \end{aligned}$$

Conversely, if the seller is willing to accept at t then

$$(29) \qquad \lim_{\epsilon \to 0} \frac{B(t+\epsilon;\bar{v}) - B(t;\bar{v})}{\epsilon} \le [B(t;\bar{v}) - \mathcal{U}^\circ(u \mid u,\bar{v})] \cdot \lim_{\epsilon \to 0} \frac{Q_t(\epsilon \mid u,\bar{v})}{\epsilon}.$$

Thus, an equilibrium requires that $u = u^\circ(t;\bar{v})$ only if

$$(30) \qquad \begin{aligned} \frac{\partial B(u;\bar{v})}{\partial u} \equiv \frac{B'(t;\bar{v})}{u^{\circ\prime}(t;\bar{v})} &= [B(t;\bar{v}) - \mathcal{U}^\circ(u \mid u,\bar{v})]\tilde{F}_t'(u \mid u;\tilde{u}\ge u,\bar{v}) \\ &= [B(u,\bar{v}) - \mathcal{U}^\circ(u \mid u,\bar{v})]\phi(u;\bar{v}). \end{aligned}$$

At this time one conditions $\tilde{F}$ on $\tilde{u} \geq u$ since the seller infers that his valuation is the smallest among the sellers; hence,

$$\lim_{\epsilon \to 0} \frac{Q_t(\epsilon \mid u, \bar{v})}{\epsilon} = \tilde{F}_t'(u \mid u; u, \bar{v}) u^{\circ\prime}(t; \bar{v}) = \phi(u; \bar{v}) u^{\circ\prime}(t; \bar{v}) .$$

The differential equation (30) is therefore the same as (17), as required. A sequential equilibrium requires that this differential equation for B must be satisfied; moreover, the boundary condition $B(\bar{v}, \bar{v}) = \bar{v}$ must be satisfied as we mentioned earlier, so the solution is given by (18). If $\bar{v} = v^*(t)$ then (30) can be interpreted as an equation for u whose solution is $u = u^{**}(t)$.

That (30) is a sufficient condition for a solution of the seller's optimization problem is assured since B is an increasing function of time. In the alternative case that $\bar{v} \leq u$ no bid from the buyer will exceed the seller's valuation and clearly the seller's optimal strategy is never to accept in this subgame.

This entirely characterizes the behavior along the equilibrium path during an ascending Dutch auction by a buyer. It is in fact the standard condition for a sequential equilibrium of an ordinary Dutch auction, and since the sellers' valuations are affiliated and $\mathcal{U}^\circ(u \mid u, \bar{v})$ is increasing with u it is known to be sufficient as well as necessary; cf. Milgrom and Weber [1982].

It is worth mentioning that our specification of the strategies during the second phase is substantially motivated by the need to be explicit in the construction. It would be adequate for some purposes to allow greater generality, requiring only that, say, $B(u, \bar{v})$ is the price at which a buyer with the valuation $\bar{v}$ expects to trade with a seller with the valuation u , rather than tying it to the specific temporal sequence of serious bids $B(\cdot; \bar{v})$ made by the buyer. In this case it would be sufficient merely to specify the seller's strategy in the form that he plans to accept the first bid exceeding $B((u^\circ)^{-1}(u); \bar{v}) \equiv P(u, \bar{v})$.

Off the Equilibrium Path

During the second phase there are several ways that play can depart from the equilibrium path. In all of these we follow a principle developed by Cramton

[1984] in more detail than we shall do here. The principle is that non-serious offers are ignored, whereas new serious offers are interpreted as equilibrium offers according to revised probability assessments by the other traders. As we shall see, this specification is sufficient to induce all traders to adhere to the equilibrium strategies. Moreover, it enables an economy of presentation by allowing that behavior in any circumstance can be determined by reference to the equilibrium strategies in that circumstance according to the traders' revised beliefs. We consider only the case of an ascending Dutch auction by a buyer; the case of a descending Dutch auction by a seller is analogous.

We first consider the buyers' behavior, and begin with the buyer conducting the Dutch auction. He can deviate by bidding either lower or higher than the equilibrium strategy requires. If he bids lower then this is a non-serious offer and the other traders expect him to revert to the equilibrium strategy; clearly he has nothing to gain by this deviation. If he bids higher then the other traders infer that his valuation is higher than previously they estimated and they expect him to continue with the sequence of bids corresponding to this higher valuation, and in particular to continue raising all the way to this higher valuation if necessary to make a transaction; in addition, subsequent reversion to his equilibrium strategy will be interpreted as non-serious, so a deviant higher bid represents a permanent commitment. Again it is fairly clear that this deviation is disadvantageous (for example, if the initial deviant bid is accepted then he has missed a chance of transacting at the lower price specified by the equilibrium strategy), but we refer to Cramton [1984; pp. 119 -- 122] for additional details.

Other buyers who intervene with bids less than the auctioneer's bids gain no advantage since their bids are non-serious. A higher bid, however, is serious and the other traders immediately infer that the intervening buyer has a correspondingly higher valuation and expect him to continue with the equilibrium strategy. Along the equilibrium path such a buyer actually has a lower valuation and, by a repetition of the previous argument, sees such an intervention as disadvantageous. This is true even if he fails to trade in this sub-

game, since in a subsequent subgame he will be treated as having the higher imputed valuation. [7]

The previous paragraph describes only one of the possible specifications. Another is that the auctioneer's strategy is actually of the form that he bids the maximum of the equilibrium bid and any intervening serious bids by other buyers, up to the limit of his reservation price. With this specification an intervener with a lower valuation has no chance of making a transaction in this subgame, but jeopardizes his terms of trade in any ensuing subgame.

Among the sellers an ask price above the auctioneer's bid is non-serious and is ignored, while an ask equal to or less than the bid can not be more advantageous than accepting the bid. Alternatively, one can simply specify as part of the rules that an ask no greater than an offered bid is construed as an acceptance of the bid.

All of these disequilibrium specifications are innocuous except for the key feature that a new higher serious bid induces other traders to revise their assessment of the auctioneer's valuation, and thereafter he is unable to lower this assessment by reducing his bids; that is, the auctioneer is essentially 'locked in' by the expectations of the other traders. This is the essential determinant of the form of the equilibrium derived by Cramton, since in anticipation of this feature in the second phase, delay in making a serious offer in the initial phase can function effectively as a signal of a trader's valuation.

Strategies in the Initial Phase

On the Equilibrium Path

Consider a seller with the valuation u contemplating a first serious offer at a time $t < S(u)$ in the initial phase. If he offers the serious ask price $p(t)$ then other traders infer that his valuation is $\check{u}(t) = S^{-1}(t) \equiv u^*(t)$ and expect that the subsequent ask prices $A(\cdot;\check{u}(t))$ will decline towards $\check{u}(t)$ at time 1. A buyer with the valuation v, therefore, plans to accept the first ask that is $A((v^\circ)^{-1}(v);\check{u}(t)) \equiv P(\check{u}(t),v)$ or less. Anticipating this behavior, the seller's

expected payoff is

$$\mathcal{E}_t\{\int_{u\sqcup\check{u}(t)}^{v^*(t)} P(\check{u}(t),\bar{v})\,d\bar{F}(\bar{v}\mid u,\tilde{u})\mid \tilde{u}{\ge}u^*(t)\} \tag{31}$$

if he conducts the Dutch auction, but of course stops at his valuation $u > \check{u}(t)$, and if no other seller intervenes. [8] Midway in the Dutch auction he could revert to his equilibrium strategy $A(\cdot;u)$ but these would be non-serious offers: the key point is that having allowed the inference that his valuation is $\check{u}(t) < u$ the seller has no later opportunity to induce other traders to revise their assessments upwards; thus, (31) represents the best that he can achieve in the second phase. If another seller intervenes this seller gets the expected payoff in the ensuing subgame with his valuation assessed to be $\check{u}(t)$.

His only other alternative (other than the trivial one of offering an ask less than $p(t)$, which is clearly disadvantageous), is to delay making a serious offer for an interim period, which we take to be infinitesimal to preserve some brevity in the notation. Compared to the first alternative this option obtains an expected gain of

$$\begin{aligned}\mathcal{G}(t;u) &= \Delta(t)\bar{F}_t'(v^*(t)\mid u;\tilde{u}{\ge}u^*(t))\cdot|v^{*\prime}(t)| \\ &\quad+\int_{u\sqcup\check{u}(t)}^{v^*(t)} \frac{\partial P(\check{u}(t),\bar{v})}{\partial\check{u}}\,d\bar{F}_t(\bar{v}\mid u;\tilde{u}{\ge}u^*(t))\cdot\check{u}'(t),\end{aligned} \tag{32}$$

where $\bar{F}_t'(v^*(t)\mid u;\tilde{u}{\ge}u^*(t))$ is the probability that in the interim the highest valuation buyer makes a first serious bid $q(t^+)$. Note here that delaying enables the seller to improve the other traders' assessment $\check{u}$ of his valuation, and this helps in the second phase. Offsetting this gain, however, is the expected loss from the chance that another seller (with the valuation $\tilde{u} = u^*(t)$) will make the first serious offer:

$$\mathcal{L}(t;u) = \int_{u\sqcup\check{u}(t)}^{v^*(t)} [P(\check{u}(t),\bar{v}) - \mathcal{U}^\circ(u\mid u^*(t),\bar{v})]\tilde{F}_t'(u^*(t)\mid u,\bar{v})\,d\bar{F}_t(\bar{v}\mid u)\cdot u^{*\prime}(t). \tag{33}$$

Thus, at time t the seller with the valuation u prefers to wait an infinitesimal period if $\mathcal{G}(t;u) > \mathcal{L}(t;u)$. An equilibrium requires, therefore, that $\mathcal{G}(t;u) \ge \mathcal{L}(t;u)$ for all times $t < S(u)$, and at the prescribed time $t = S(u)$ for his first serious offer that $\mathcal{G}(S(u);u) \le \mathcal{L}(S(u);u)$.

Recall now that $\breve{u}(t) \equiv u^*(t)$ and therefore $\breve{u}'(t) = u^{*\prime}(t)$; consequently, the condition derived here that, say, $\mathcal{G}(S(u);u) = \mathcal{L}(S(u);u)$ in equilibrium is equivalent to the optimality condition (24) in the corresponding revelation game if and only if $|v^{*\prime}(S(u))| = u^{*\prime}(S(u))$. This, then, is the missing condition that knits together the equilibrium by connecting the time parameterizations of the buyers and sellers during the initial phase. This condition must hold at all times that a first serious offer might be made by either a seller or a buyer. Hence, for an equilibrium such as we have specified in which all times have the potential for first serious offers, it is necessary that

$$v^{*\prime}(t) + u^{*\prime}(t) = 0 \tag{34}$$

at each time $t \in (0,1)$.

Off the Equilibrium Path

Deviations from the equilibrium path in the initial phase can be of the following types. A trader can make a serious offer too early or too late, in which case our analysis for off-the-equilibrium-path behavior in the second phase applies: too early saddles the trader with other traders' too-favorable assessment of his valuation; and too late, as we have just seen, foregoes a profitable opportunity. A trader can make a serious offer that is better than the expected serious offer, but this is clearly disadvantageous. Lastly, when another trader, say a seller, makes a serious offer then a buyer can not accept though the equilibrium strategy says he should; again, by construction, this is unprofitable. [9]

Construction of the Equilibrium Strategies

We now show how the various conditions that have been derived combine to determine each of the functions that specify the equilibrium strategies.

First, one can determine $\hat{u}$ and $\hat{v}$ by invoking (24) at $u = \hat{u}$, and its analogue for a buyer at $v = \hat{v}$. For brevity we display only the seller's condi-

tion:

$$
(35)\qquad \begin{aligned} 0 = \int_{\hat{u}}^{\hat{v}} &\left\{ [U^\circ(\hat{u} \mid \hat{u}, \bar{v}) - \hat{u}]\phi(\hat{u};\bar{v}) + \frac{\partial A(\hat{u};\hat{u})}{\partial u} \right\} d\bar{F}(\bar{v} \mid \hat{u}; \tilde{u} \geq \hat{u}, \bar{v} \leq \hat{v}) \\ &+ [\hat{v} - \hat{u}]\bar{F}'(\hat{v} \mid \hat{u}; \tilde{u} \geq \hat{u}, \bar{v} \leq \hat{v}), \end{aligned}
$$

using $P(\hat{u},\bar{v}) = A(\hat{u},\hat{u}) = \hat{u}$ for $\bar{v} \geq v^\circ(1;\hat{u}) = \hat{u}$, $\mathbf{\Delta}(1) = \hat{v} - \hat{u}$, etc. Of course this condition and its analogue for the buyer are to be solved for the solution with $\hat{v} > \hat{u}$ rather than the trivial solution $\hat{v} = \hat{u}$, as mentioned in §3.

Next, as in Figure 1, if it is u^* that is continuous at $t = 0$, then it suffices to parameterize time in the initial phase so that

$$
(36)\qquad \begin{aligned} u^*(t) &= u^* + t \cdot [\hat{u} - u^*], \qquad & S(u) &= \frac{u - u^*}{\hat{u} - u^*}; \\ v^*(t) &= \hat{v} + [1 - t] \cdot [\hat{u} - u^*], \qquad & T(v) &= 1 - \frac{v - \hat{v}}{\hat{u} - u^*}; \end{aligned}
$$

which ensures that $v^{*\prime}(t) + u^{*\prime}(t) = 0$. Note that the magnitude of the discontinuity at $t = 0$ is

$$
(37)\qquad v^* - v^*(0^+) = [v^* - \hat{v}] - [u^* - \hat{u}];
$$

if this were negative one would parameterize so that $v^{*\prime}(t) = \hat{v} - v^*$, etc. [10]

The remaining step in the construction of the initial phase is to determine $u^{**}(t)$ and $v^{**}(t)$, and this is done by invoking (24) again for the seller and the analogous condition for the buyer. Without displaying the long formulas, observe that (24) for the seller depends on $v^{**}(t) \equiv v^\circ(t;u)$ for $t = S(u)$ via (25).

We presented in (18) and (21) the construction of the reduced forms of the buyers' and sellers' strategies in the second phase. To obtain extensive-form representations for, say, a seller's Dutch auction it suffices to adopt the convenient parameterization of time in which the offers $A(\cdot;u)$ decline at a constant rate from $A(u, v^{**}(S(u)))$ down to u over the interval $S(u) < t < 1$:

$$
(38)\qquad A(t;u) = \frac{1 - t}{1 - S(u)} A(u, v^{**}(S(u))) + \frac{t - S(u)}{1 - S(u)} u.
$$

One can then solve for the buyer's acceptance strategy $v^{\circ}(\cdot;u)$ by using (21) and the condition that $A(t;u) = A(u, v^{\circ}(t;u))$.

These constructions all depend on the traders' expected payoffs $\mathcal{U}^{\circ}$ and $\mathcal{V}^{\circ}$ in ensuing subgames with one less seller and one less buyer. Thus the construction actually begins with the case in which there is a single seller or a single buyer and then proceeds inductively to compute the strategies and expected payoffs in markets with successively larger numbers of sellers and buyers. In the next section we address the special case of the 'endgame' market in which there is only one remaining active trader on one or both sides of the market.

5. The Endgames

We first consider the case of an endgame in which there is a single seller and several buyers; the case of a single buyer and several sellers is analogous. This case is degenerate in that there is no competitive pressure on the seller that determines his strategy. Referring to (33) one sees that in the initial phase the seller suffers no loss from delaying a serious offer, since there is no chance that another seller intervenes with a serious offer in the interim. Similarly, in the second phase after a serious offer by a buyer the seller again incurs no loss from delay, as indicated by (28), and prefers to wait for the buyer's ask to decline. Further, since each buyer sees no chance that the seller will make a first serious offer in the initial phase, and expects that delay will not function as a signal to improve his terms of trade with the seller in the second phase, a buyer obtains no gain from delay. He does, however, expect to lose by delay since there is a chance that another buyer will capture the opportunity to trade with the one remaining active seller. [For these conclusions one can examine the analogues of (30) and (33) for a buyer.] After a serious bid is offered, moreover, other buyers in the second phase are not deterred from intervening; thus the serious bids in the second phase are immediately driven to the second-highest valuation among the buyers. Thus an apparent extension of the specified equilibrium to the endgame has the following scenario: all the buyers open with serious offers, and immediately the maximum

serious bid is driven up to the second-highest valuation. That is, the endgame is much like an ascending English auction, but compressed into the first instant.

This is an unsatisfactory model of the endgame unless it is elaborated further.[11] It seems better to dispense with the continuity of time at the initial moment, since all the action takes place there, and to model explicitly the trading process in finer detail. We have the benefit of the results obtained by Milgrom and Weber [1982; Theorems 11, 15, and 13]: the seller prefers an English auction to a Dutch auction (and to several variants), and the seller prefers to reveal his private information.[12] These results permit an endgame specification that is consistent with the specification of the equilibrium for the other subgames, although fanciful in one aspect. If the seller reveals his valuation u and the buyers bid in an ascending English auction then the endgame's expected payoff for a buyer with the valuation v is $\mathcal{V}(v) = \mathcal{E}\{v \sqcap (u \sqcup \tilde{v}) \mid v\}$, and the seller's expected payoff is $\mathcal{U}(u) = \mathcal{E}\{u \sqcup v_{[2]} \mid u\}$, where $v_{[2]}$ is the second-highest among the buyers' valuations. The fanciful aspect, of course, is the source of the credible signal that his valuation is u. From Milgrom and Weber we know that if the seller can signal credibly that his valuation is u then he will do so, but the present formulation does not include any such signalling mechanism in the absence of competitive pressure on the seller. To allow the possibility of credible signalling in the absence of competitive pressure, given that it is in the seller's interest, it suffices to introduce an auxilliary feature. The simplest device is to allow a final instant in which the seller can either accept the outstanding bid or ask a final take-it-or-leave-it price before the market closes. Another alternative is to introduce an additional source of impatience, as described below. In any case one expects that whatever refinement is used to resolve the indeterminacy in the endgame will have only a slight effect on the construction of the specified strategies in the earlier subgames with more numerous traders.[13]

An additional degeneracy is introduced if the endgame involves only one seller and one buyer, corresponding to the case that the market opened orig-

inally with equal numbers of sellers and buyers ($m = n$). This case is essentially one of pure bargaining, and there is no formulation that is directly consistent with the construction adopted for the subgames with more traders in which competitive pressure determines the signalling mechanism. One could, of course, adopt the expedient of assuming that the traders split the gains from trade, if any, according to some maintained hypothesis. For example, if the endgame consists only of division of the gains according to an offer by one party that is either accepted or rejected by the other, then a focal-point equilibrium suffices and any one can be specified as the common-knowledge expectation of both traders: this is essentially the model studied in the bargaining experiments by Roth and Schoumaker [1983].

A preferable model allows that a positive rate of interest makes the traders impatient for conclusion of a transaction, as in the bargaining model of Cramton [1984]. In this case, impatience resurrects delay as an effective signal of a trader's valuation.

We do not present here the alterations in the construction entailed by a positive interest rate, but refer the reader to the exposition by Cramton [1984] for the case of one buyer and one seller. The main conclusion is that if the interest rate is positive then the endgames are quite like all the other subgames and require no special treatment. Admittedly this is not fully satisfactory for studies of experiments in which the duration of the market, typically measured in minutes or hours, is too short to make plausible values of the interest rate an important determinant of the traders' strategies, but it has the advantage of unifying the theory of the endgames with the other subgames in the construction. Optionally, one could interpret the interest rate in the endgame as infinitesimal compared to the competitive pressure (i.e., the hazard rates) in the earlier subgames, or in an experimental situation one could interpret the interest rate as a generic impatience (e.g., fatigue) with behavioral origins. In some experimental designs the time of the closing of the market is uncertain and in this case the hazard rate of termination serves the same role as an interest rate.

In general, any source of impatience suffices for delay to be an effective signal, and here we have concentrated on the role of competitive pressure as a source of impatience, so the endgames necessarily present significant degeneracies.

6. Remarks

The sequential equilibrium proposed and partially verified here is likely only one of many possibilities. Its form derives primarily from the presumed role of delay in making or accepting a serious offer as the sole signal of a trader's valuation. Underlying the construction is Cramton's key distinction between serious and non-serious offers, the special character of the off-the- equilibrium-path beliefs, and the Markov-perfect character of the on-the-equilibrium-path beliefs. One can reasonably conjecture that there are many other equilibria that accomplish signalling by different mechanisms; e.g., by using history-dependent strategies. The one presented here is interesting mainly because it invokes delay to exploit directly the temporal features of the trading process. It is, moreover, relatively simple and tractable to analyze, particularly since we can draw on the previous insights of Cramton for the special case of one seller and one buyer. For purposes of comparison with experimental results it is a useful first step in providing a testable model of equilibrium behavior.

The deficiencies of the model for use in experimental work are severe nevertheless. Most of the experiments that have been conducted allow that each trader may demand or supply several items, the subjects could plausibly be supposed to be risk averse, etc. The crucial deficiencies, however, are inescapable consequences of the game-theoretic formulation. These are, first, that the probability distribution of the traders' valuations is common knowledge (which is rarely controlled in experiments); and second, that the subjects are able to know or compute equilibrium strategies and select one equilibrium in a way that is common knowledge (including, for example, the parameterization of time). Ledyard [1984] emphasizes that nearly any undominated

strategies can be justified as equilibrium behavior if there is no control on the risk aversion of the traders. Easley and Ledyard [1983], moreover, describe simple behavioral (non-equilibrium) rules-of-thumb that suffice to explain the experimental data to a substantial degree. It remains an open question, therefore, whether a game-theoretic hypothesis such as pursued here will prove to be the most useful explanation of the experimental data.

Some of the implications of our model and the specified equilibrium are, in fact, too strong to fit the data well. For example, the equilibrium predicts that traders transact in order of their valuations, and that no traders with extra-marginal valuations (e.g., sellers' valuations above the Walrasian clearing price) succeed in trading. As Easley and Ledyard [1983] report, these properties are often contradicted by experiments. One must be cautious, however, since such experiments are a test of the compound hypothesis that the common knowledge structure is the one specified, as well as the equilibrium strategies.

On the other hand, the implications of the equilibrium for theoretical studies of price formation and the micro-structure of markets are favorable. In particular, the equilibrium presented here offers a specific interpretation of the trading process that lends substance to the Walrasian model of markets when there are many traders on both sides of the market. If there are many buyers and sellers then the competitive pressures (measured by the hazard rates) drive all traders to offer and accept prices approximating their continuation values in ensuing subgames; since the gains from trade will be nearly exhausted these continuation values must approximate, say for a seller, the maximum of the Walrasian price and his valuation. That is, asymptotically $\mathcal{U}(u) \to p^\circ \sqcup u$ as $m \sqcap n \to \infty$, where p° is the asymptotic Walrasian clearing price. [14]

Bid-ask markets are familiar in commodity exchanges and it seems plausible to extrapolate from the present results that near exhaustion of the gains from trade, at prices approximating the Walrasian price, are predictable features of these markets. Even with small numbers of traders (as in the usual experimental designs), to the extent that gains from trade are nearly exhausted it is

predictable that transaction prices converge over time to values close to the Walrasian price.

The suggested equilibrium also offers a concrete explanation of the mechanism by which the dispersed information about traders' valuations is manifested in the prices at which transactions are consummated. The mechanism, according to the present hypothesis, is multilateral sequential bargaining in which the traders are endogenously matched for transactions via a signalling process using delay as the primary signal. Other signalling mechanisms may be possible, but it appears that delay suffices and therefore this provides a presumptive hypothesis from which further studies can proceed.

All of the above remarks must, of course, be taken as speculative until the theory of affiliated random variables is applied, as in Milgrom and Weber [1982] for the case of ordinary auctions, to determine whether or not the proposed equilibrium strategies also satisfy the requisite global optimality properties that would be sufficient to establish the validity of the equilibrium. The satisfaction of the necessary conditions as established here and the internal consistency of the construction do, I think, lend encouragement that an equilibrium of this form will obtain. If so, it unifies a spectrum of market structures ranging from bargaining to perfect competition.

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Footnotes

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† I gladly dedicate this chapter to my friend and colleague Kenneth Arrow. Over a span of twenty years I have benefitted continually from Ken's unbounded creativity, sparkling intellect, and professional leadership, as well as learning from the finest example the humane concerns of the scholar. This chapter is a small token of my great appreciation; I wish only that it achieved the definitive solution so often attained in Ken's work.

1. See also Rubinstein [1985] for a related model with one-sided incomplete information about one party's discount factor.

2. See also the related results by Chatterjee and Samuelson [1984].

3. Apologies are offered for the complex notation, most of which stems from the feature that traders' reservation prices are not assumed to be independent: the generality is necessitated by the fact that even if independence were assumed initially it would be lost as trading proceeded, as evidenced by equation (5) below.

4. For example, it suffices that the valuations are stochastically independent and the sellers' valuations are identically distributed and so are the buyers' valuations. Affiliation also allows positive correlation among the valuations. If the proposed equilibrium is to be verified it seems certain that affiliation will prove to be the relevant sufficiency condition, judging from the results established by Milgrom and Weber [1982].

5. This definition of consistency is loosely stated; see Kreps and Wilson [1982] for a complete statement. In the present context the only null events involves histories that have zero probability according to the strategies. For a formal definition of a conditional probability system in this context see Myerson [1984].

6. If the buyers' valuations happen to be independent with the distribution function H then $\bar{F}'_{S(u)}(\bar{v} \mid u; u) = nH'(\bar{v})/H(v^*(S(u)))$.

7. Such an ensuing subgame is not of the form initially assumed, since the buyer's valuation is assumed (incorrectly) to be known. We address this exception in the obvious way by specifying that in the ensuing subgame he is expected to open immediately with a serious bid.

8. Actually one could allow that he stops at some higher ask, essentially turning the auction over to another seller or buyer, and gets the expected payoff in an ensuing subgame, but this option plays no role subsequently so we omit the corresponding notational refinements.

9. As usual in games of timing, as here in the initial phase, the disequilibrium analysis is essentially trivial, since delay is the only signal with informational content and each disequilibrium action has an equilibrium interpretation.

10. Alternatively one can retain the parameterization (36) and instead set $v^*(t) = v^*$ for $t < t^\circ \equiv 1 - [v^* - \hat{v}]/[\hat{u} - u^*]$. In this case, only the sellers signal their valuations by delay in the initial interval $[0, t^\circ]$. In one way this is a preferable specification since it reduces the reliance of the equilibrium on the tie-breaking rule.

11. For comparisons with experimental results, one should note that the exact formulation of the endgame is immaterial to the extent that it is unlikely that the market will arrive at an endgame before the gains from trade are exhausted or time expires.

12. Their Theorems 18 and 19 do not apply here since a reserve price other than the seller's valuation is inconsistent with a sequential equilibrium.

13. If the buyers' valuations happen to be independent in the endgame then the

following device will also suffice, since in this case a Dutch auction is equivalent to an English auction for the seller: the seller conducts a Dutch auction, as in (21), and the buyers infer his valuation from the rate at which his ask prices decline.

14. For example, if the sellers' and buyers' valuations are distributed independently according to the distribution functions G and H then $G(p^\circ) + H(p^\circ) = 1$.

FIGURE 1. **Equilibrium Strategies in the Initial Phase**

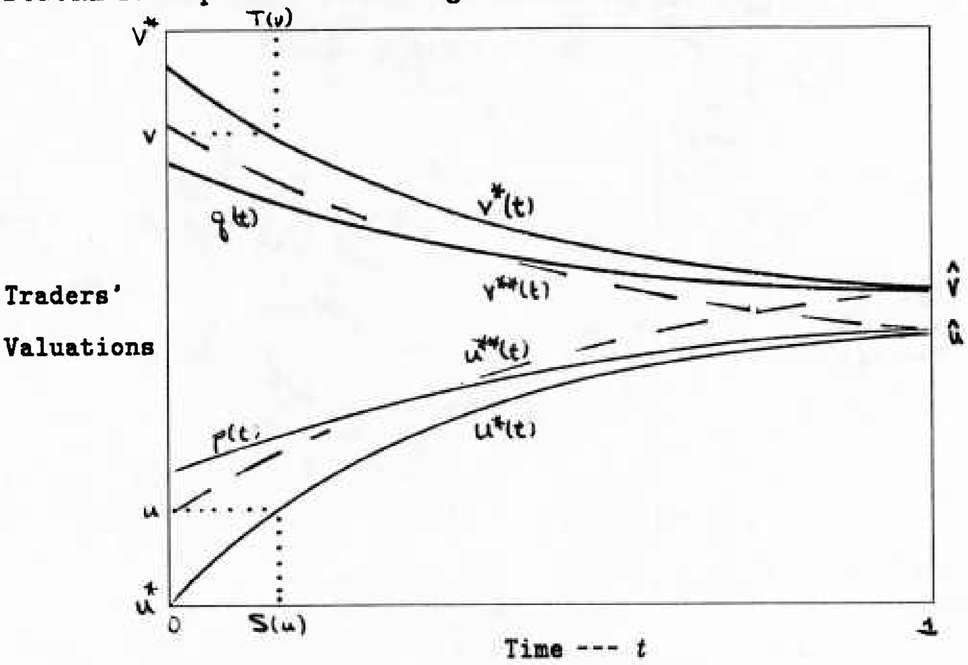


FIGURE 2. **Equilibrium Strategies in the Second Phase**

After a First Serious Ask by a Seller.

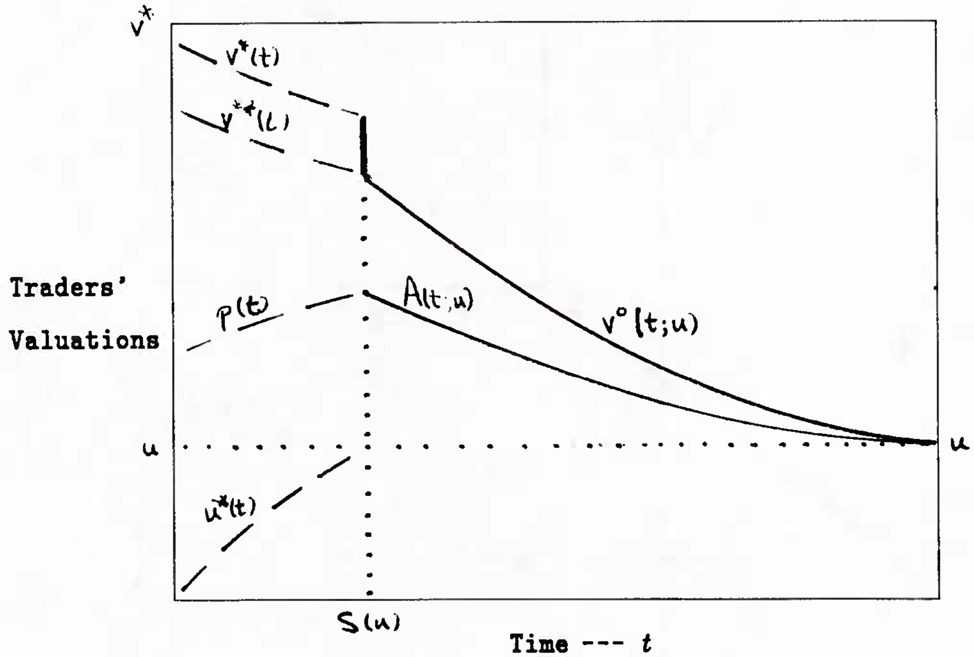


FIGURE 3. **Equilibrium Strategies in the Second Phase After a First Serious Bid by a Buyer.**

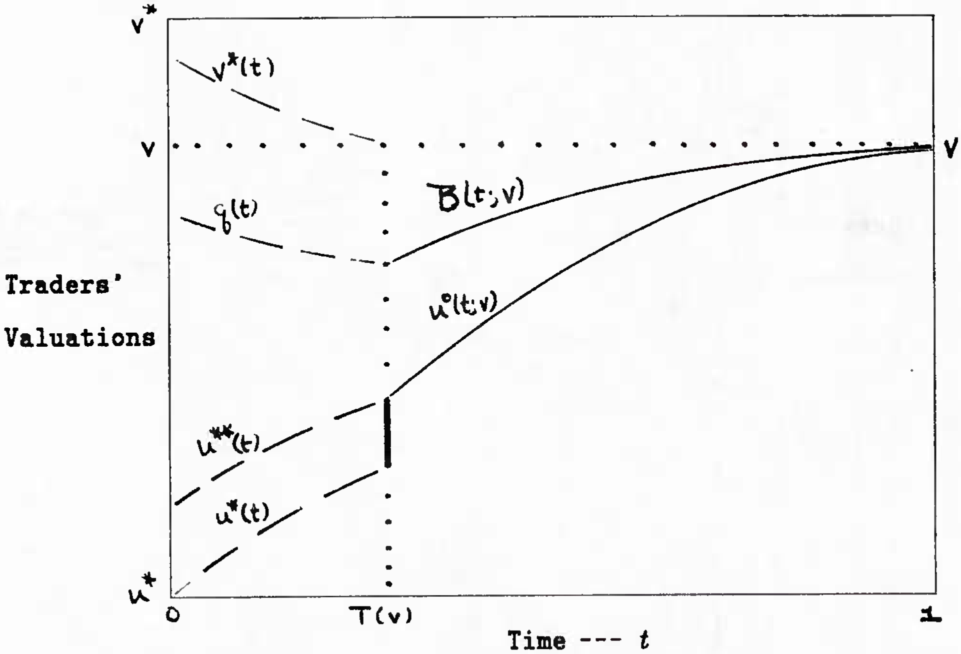


FIGURE 4. **Subgame Payoffs for a Seller if** $u < \hat{u}$.

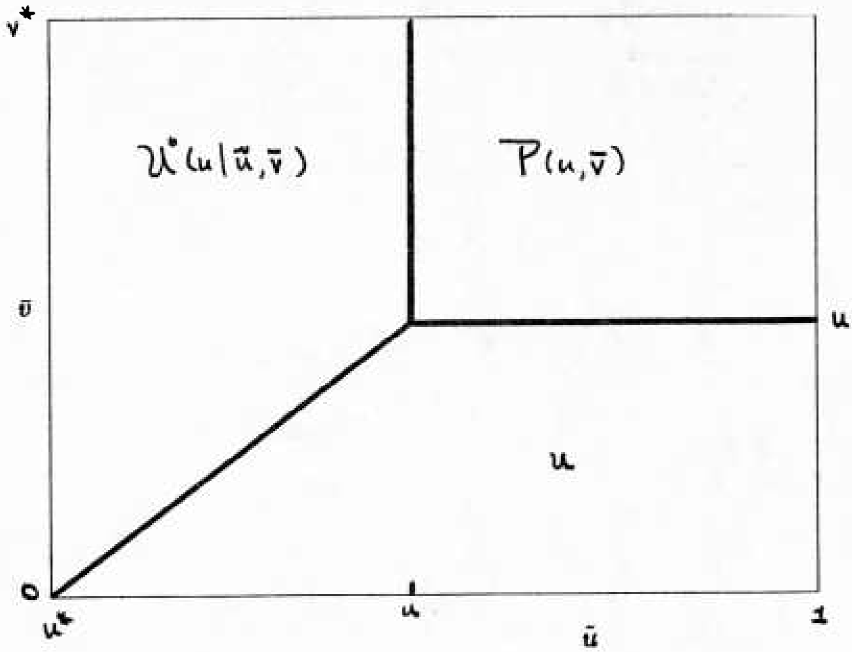


FIGURE 5. **Subgame Payoffs for a Seller if $\hat{u} \leq u \leq \hat{v}$.**

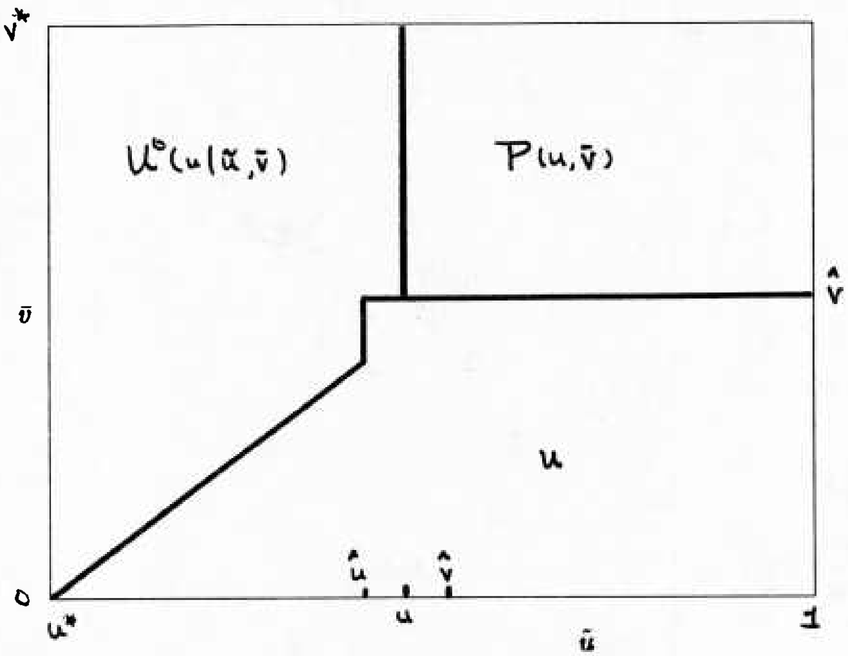


FIGURE 6. **Subgame Payoffs for a Seller if $\hat{v} < u$.**

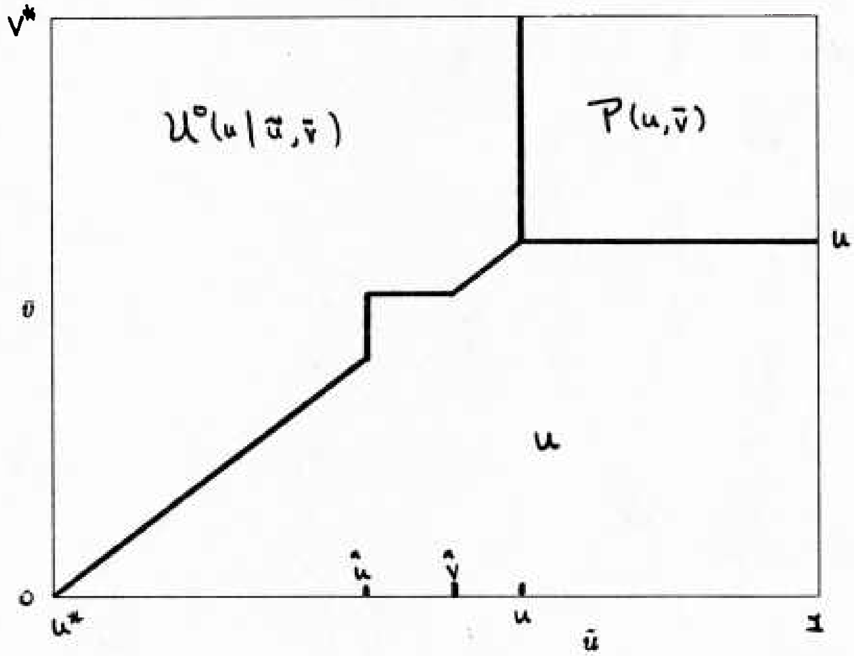
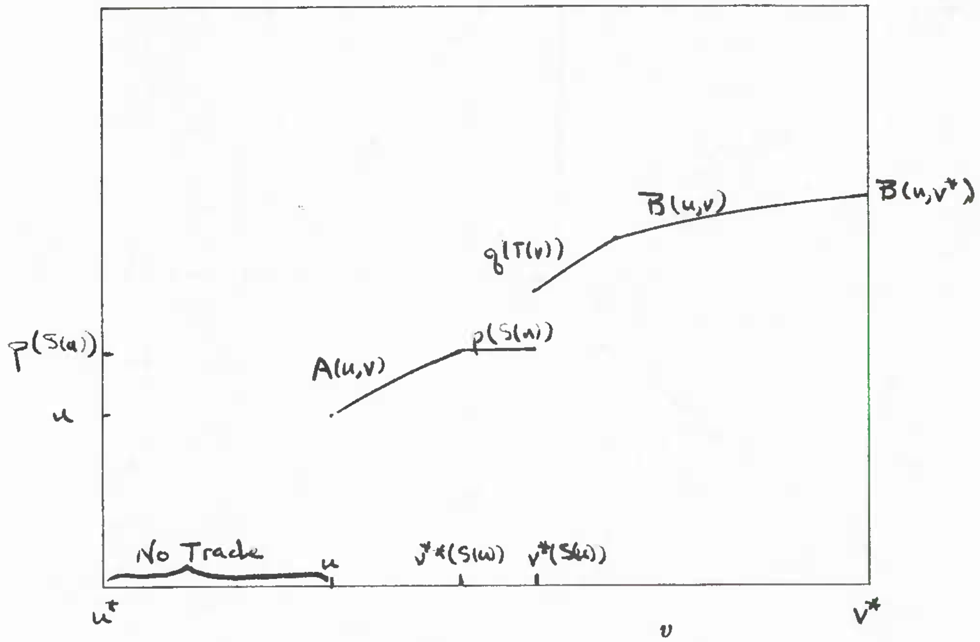


FIGURE 7. **The Transaction Price** $P(u,v)$



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